AIAA JOURNAL
Vol. 33, No. 4, April 1995

Highly Accurate Modal Method for Calculating Eigenvector Derivatives in Viscous Damping Systems

Qing-Hua Zeng*
Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China

A highly accurate modal superposition method for computing complex eigenvector derivatives in viscous damping systems has been developed. When higher modes are truncated, the conventional modal superposition method cannot give an accurate solution, and the errors may become significant. In this paper, calculating the derivatives is regarded as calculating the structural response to harmonic exciting. Using multiple modal accelerations and shifted-poles, highly accurate results would be obtained when only few modes are used. All of the available modal superposition methods would be directly obtained from the proposed method. Numerical examples show that it achieves better calculation efficiency than all of the available modal methods and Nelson's method when more than one eigenvector derivative is of interest. Moreover, the presented method can be used to improve response calculations and substructure syntheses.

Introduction

EIGENVECTOR derivatives with respect to system parameters are widely used in structural design, structural stability analysis, finite element model updating, and structural control. Since the earlier work of Fox and Kapoor,[1] many methods have been developed.[2,3] All of the available methods could be categorized into three groups: modal method,[1,3,4] direct method,[1,4] and iteration method.[5,6] Most of the iteration methods suffer from slow convergence rates and are less efficient. Recently, Ting[6] improved the effectiveness of the method. Because the direct methods give out the exact solutions, they are widely used. However, when the derivatives of a large number of eigenvectors are demanded, the calculation is very time consuming. Here, the modal methods are more suitable.

The modal methods employ a modal superposition idea; therefore, the accuracy is dependent on the number of modes used in calculation. When the complete set of eigenvectors is used, the exact solution could be obtained. To guarantee the accuracy, the classical modal method[1] needs higher eigenvectors. Recently, Wang[4] presented a modified modal method, in which a modal acceleration type approach was used to approximate the contribution of truncated higher modes. Therefore, the modified modal method converges faster than the classical method. Ma and Hagiwara[7]). improved the modified method with an appropriate shift value, better convergence than that of the modified method is achieved, and systems having zero eigenvalues can be directly analyzed. To further improve the convergence, the improved modified modal method[8,10] and accurate modal superposition method[9] have been proposed by several researchers. The basic original idea of the aforementioned improved modal methods was found in the Hu's book.[11] The idea was used to improve the convergency of dynamic flexibility in series form for no damping systems.

All of the aforementioned methods were used for real eigenvectors. For the cases of complex eigenvectors in viscous damping systems, the three groups of methods were presented.[12,13] The classical modal method for the complex eigenvectors, which was similar to the method presented in Ref. 1 for real eigenvectors, suffers from heavy modal truncation errors. Recently, Akgun[14] presented a new family of modal methods for the calculation of the derivatives in non-self-adjoint systems. The modal acceleration approach was used to improve the convergence. In this paper, calculating the derivatives in viscous damping systems is regarded as calculating a structural response to a harmonic exciting, using multiple modal accelerations and shifted-poles; a highly accurate modal method for calculating the derivatives has been developed. There are clear mathematical and physical meanings in conducting the proposed method, from which all of the available methods would be directly obtained. Furthermore, fewer eigenvectors (even only the specific eigenvector that is being considered) are required for the predetermined accuracy. As a result, the method is more efficient in computation than Akgun's method and Nelson's method when more than one eigenvector derivative is of interest. Numerical examples show that the proposed method is correct and drastically improves solution accuracy. The method can directly be applied in systems with zero eigenvalues. Although the method is developed to calculate eigenvectors derivatives, it can be used to improve response calculations and substructure syntheses.

Analysis

Governing Equations

The general equation of motion for an N degree-of-freedom (DOF) system with viscous damping and excitation is

$$M\ddot{x} + C\dot{x} + Kx = g(t) \tag{1}$$

where M, K, and C are $N \times N$ mass, stiffness, and damping matrices, respectively. The matrices are real, symmetric, and positive definite. Equation (1) can be rewritten in the following form,

$$A\dot{y} + By = h(t) \tag{2}$$

where

$$A = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix},$$

$$y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \qquad h(t) = \begin{bmatrix} g(t) \\ 0 \end{bmatrix}$$

Consider $g(t) = 0$ and general trial solution $y = \Phi e^{st}$, and we obtain the eigenequation

$$(sA + B)\Phi = 0 \tag{3}$$

The N pair complex conjugated eigensolutions (eigenvalues s_j and s_j^*, and eigenvectors Φ_j and Φ_j^*, $j = 1, 2, \ldots, N$) are determined

Received March 29, 1994; revision received Oct. 3, 1994; accepted for publication Oct. 3, 1994.

*Associate Professor, Institute of Vibration Engineering.

by the solution of Eq. (3). A modal matrix ψ is formed by the eigenvectors, and orthogonality properties may be expressed as

$$\psi^T A \psi = \begin{bmatrix} \ddots & & \\ & a_j & \\ & & \ddots \end{bmatrix} \tag{4a}$$

$$\psi^T B \psi = \begin{bmatrix} \ddots & & \\ & b_j & \\ & & \ddots \end{bmatrix} \tag{4b}$$

$$\Phi_j^T A \Phi_j = a_j \tag{4c}$$

$$s_j = -\frac{b_j}{a_j} \qquad j = 1, 2, \ldots, N \tag{4d}$$

Simply writing s_j as s, Φ_j as Φ, and a_j as a and taking derivative of Eq. (3) with respect to system parameter p yield

$$\frac{\partial s}{\partial p} A\Phi + s\frac{\partial A}{\partial p}\Phi + \frac{\partial B}{\partial p}\Phi + (sA + B)\frac{\partial \Phi}{\partial p} = 0$$

The following two equations can be sequentially derived

$$\frac{\partial s}{\partial p} = -\left[\Phi^T\left(s\frac{\partial A}{\partial p}\Phi + \frac{\partial B}{\partial p}\right)\Phi\right]\bigg/ a - \frac{\Phi^T(sA + B)}{a}\frac{\partial \Phi}{\partial p} \tag{5}$$

$$(sA + B)\frac{\partial \Phi}{\partial p} = -\left(\frac{\partial s}{\partial p}A + s\frac{\partial A}{\partial p} + \frac{\partial B}{\partial p}\right)\Phi \tag{6}$$

Let

$$\mathrm{d} = \frac{\partial \Phi}{\partial p}$$

$$f = -\left(\frac{\partial s}{\partial p}A + s\frac{\partial A}{\partial p} + \frac{\partial B}{\partial p}\right)\Phi \tag{7}$$

Equation (6) is simply rewritten as

$$(sA + B)\mathrm{d} = f \tag{8}$$

From Eqs. (3) and (4a), the derivative of the jth eigenvalue s_j can be expressed as

$$\frac{\partial s_j}{\partial p} = \left[-\Phi_j^T\left(s_j\frac{\partial A}{\partial p} + \frac{\partial B}{\partial p}\right)\Phi_j\right]\bigg/ a_j \tag{9}$$

From Eq. (9) the derivatives of the jth modal frequency and damping ratio can be calculated. In no repeated eigenvalues cases, the rank of the matrix $(s_j A + B)$ is $2N - 1$. To solve Eq. (8), the following equation, which is derived by taking the derivative of Eq. (4c), should be supplemented

$$\Phi_j^T A d_j = -\tfrac{1}{2}\Phi_j^T\frac{\partial A}{\partial p}\Phi_j \tag{10}$$

Modal Superposition and Modal Acceleration

Comparing Eq. (8) with Eq. (2), we find out that computing the derivative d by solving Eq. (8) is equivalent to computing the response to harmonic excitation fe^{st}. With the modal superposition approach, we obtain

$$\mathrm{d} = \sum_{k=1}^{N}\left[\frac{\Phi_k\Phi_k^T/a_k}{s - s_k} + \frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s - s_k^*}\right] f \tag{11}$$

When s approaches the limit s_j, the derivative of the jth eigenvector can be presented as

$$\mathrm{d}_j = \sum_{k=1,k\neq j}^{N}\left[\frac{\Phi_k\Phi_k^T/a_k}{s_j - s_k} + \frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s_j - s_k^*}\right] f_j + \lim_{s\to s_j}\left[\frac{\Phi_j\Phi_j^T/a_j}{s - s_j} + \frac{\left(\Phi_j\Phi_j^T/a_j\right)^*}{s - s_j^*}\right] f \tag{12}$$

where

$$\lim_{s\to s_j}\frac{\Phi_j^T f}{s - s_j} = \lim_{s\to s_j}\left[\Phi_j^T(sA + B)\frac{\partial \Phi}{\partial p}\right]\bigg/(s - s_j)$$

Because of the symmetric of the matrices A and B, the previous limitation 0/0. Using Eq. (10), the limitation is found

$$\lim_{s\to s_j}\frac{\Phi_j f}{s - s_j} = -\tfrac{1}{2}\Phi_j^T\frac{\partial A}{\partial p}\Phi_j \tag{13}$$

Substitute Eq. (13) into Eq. (12), and let

$$\mathrm{d}_{jj} = -\left(\Phi_j\Phi_j^T\frac{\partial A}{\partial p}\Phi_j\right)\bigg/ 2a_j$$

and d_j can be represented as

$$\mathrm{d}_j = \left\{\sum_{k=1,k\neq j}^{N}\left[\frac{\Phi_k\Phi_k^T/a_k}{s_j - s_k} + \frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s_j - s_k^*}\right] + \frac{\left(\Phi_j\Phi_j^T/a_j\right)^*}{s_j - s_j^*}\right\} f_j + \mathrm{d}_{jj} \tag{14}$$

The convergence of Eq. (14) is poor. For accurately calculating the derivative d_j, the higher modes are required. In a practical situation, there are only some lower modes available, and modal truncation errors will be significant. In response calculations, the modal acceleration approach[15] is used to speed up the convergence and reduce the truncation errors. Separate the response d into a pseudostatic response d_{s0} and a dynamic correction response d_{d0}

$$\mathrm{d} = \mathrm{d}_{s0} + \mathrm{d}_{d0}$$

where

$$\mathrm{d}_{s0} = B^{-1} f$$

$$\mathrm{d}_{d0} = \mathrm{d} - \mathrm{d}_{s0}$$

or

$$\mathrm{d}_{d0} = (sA + B)^{-1} f - B^{-1} f$$

Substituting Eq. (4) into the previous equation, it yields

$$\mathrm{d}_{d0} = \psi\begin{bmatrix} \ddots & & \\ & \dfrac{1}{a_k(s - s_k)}\left(\dfrac{s}{s_k}\right) & \\ & & \ddots \end{bmatrix}\psi^T f$$

Hence, the total response is

$$d = B^{-1}f + \sum_{k=1}^{N}\left[\left(\frac{s}{s_k}\right)\frac{\Phi_k\Phi_k^T/a_k}{s - s_k} + \left(\frac{s}{s_k^*}\right)\right.$$
$$\left.\times\frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s - s_k^*}\right]f \tag{15}$$

Comparing Eq. (15) with Eq. (11), the derivative becomes

$$d_j = \left\{B^{-1} + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s}{s_k}\right)\frac{\Phi_k\Phi_k^T/a_k}{s_j - s_k} + \left(\frac{s_j}{s_k^*}\right)\right.\right.$$
$$\left.\left.\times\frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s_j - s_k^*}\right] + \left(\frac{s_j}{s_j^*}\right)\frac{\left(\Phi_j\Phi_j^T/a_j\right)^*}{s_j - s_j^*}\right\}f_j + d_{jj} \tag{16}$$

When $k > j$, and $|s_j/s_k| < 1$, the convergence of Eq. (16) is better than that of Eq. (14). The effects of truncated higher modes are reduced.

Multiple Modal Accelerations

In the modal acceleration approach, the convergence of the series is speeded up through preliminary calculation of the pseudostatic response d_{s0} to the excitation f. Based on the similar idea, if the "pseudostatic" response d_{s1} to the combination force of f and inertia force that comes from the response d_{s0} is preliminarily calculated, the convergence would be further improved for including the effects of the inertia (and damping). The pseudostatic response is

$$d_{s1} = B^{-1}[f - sAd_{s0}]$$

or

$$d_{s1} = B^{-1}[I - sAB^{-1}]f$$

At this time, the dynamic correction is

$$d_{d1} = d - d_{s1}$$

Based on Eq. (4), we have

$$d_{d1} = \psi\begin{bmatrix}\ddots & & \\ & \left(\dfrac{s}{s_k}\right)^2\dfrac{1}{a_k(s - s_k)} & \\ & & \ddots\end{bmatrix}\psi^T f$$

By the similar procedure as the modal acceleration approach, the eigenvector derivative is given as

$$d_j = \left\{B^{-1}\left(I - s_jAB^{-1}\right) + \sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j}{s_k}\right)^2\frac{\Phi_k\Phi_k^T/a_k}{s_j - s_k}\right.\right.$$
$$\left.\left.+\left(\frac{s_j}{s_k^*}\right)^2\frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s_j - s_k^*}\right] + \left(\frac{s_j}{s_j^*}\right)^2\frac{\left(\Phi_j\Phi_j^T/a_j\right)^*}{s_j - s_j^*}\right\}f_j + d_{jj} \tag{17}$$

When $|s_j| < |s_k|$, there are

$$\left|\frac{s_j}{s_k}\right|^2 < \left|\frac{s_j}{s_k}\right|^1 < 1$$

and Eq. (17) converges faster than Eq. (16). Based upon the similar procedure, when multiple modal accelerations are used, we will express the jth eigenvector derivative as

$$d_j = \left\{B^{-1}\sum_{m=0}^{M_a-1}\left(-s_jAB^{-1}\right)^m\right.$$
$$+\sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j}{s_k}\right)^{Ma}\frac{\left(\Phi_k\Phi_k^T/a_k\right)}{s_j - s_k} + \left(\frac{s_j}{s_k^*}\right)^{Ma}\frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s_j - s_k^*}\right]$$
$$\left.+\left(\frac{s_j}{s_j^*}\right)^{Ma}\frac{\left(\Phi_j\Phi_j^T/a_j\right)^*}{s_j - s_j^*}\right\}f_j + d_{jj} \tag{18}$$

For a larger value of M_a, the second series in Eq. (18) converges very fast. Only using several lower modes, the highly accurate results would be obtained. The first series can be calculated in iteration. Equation (18) is similar to one presented by Akgun.[14]

Multiple Modal Accelerations with Shifted-Poles

There are two series in Eq. (18): one is the first M_a terms of $(s_jA + B)^{-1}$ expanded by Taylor's series at the center of the s plane, and the other is the expansion of $(s_jA + B)^{-1}$ by complex eigenvectors. To speed up the convergence of the second series, the value of M_a should be as large as possible. In the first series, because

$$\left(s_jAB^{-1}\right)^m = \psi^{-T}\begin{bmatrix}\ddots & & \\ & \left(\dfrac{s_j}{s_k}\right)^m & \\ & & \ddots\end{bmatrix}\psi^T$$

the series isn't convergent when $j > 1, |s_j/s_1| > 1$. The M_a couldn't be too large for the first series. Therefore, it is difficult to choose suitable M_a value for the two series. A key problem is that the expansion of $(s_jA + B)^{-1}$ at the center of the s plane isn't convergent. We should expand $(s_jA + B)^{-1}$ in Taylor's series in the region of s_j. The term $(s_jA + B)^{-1}$ is expanded in Taylor's series at the position β as

$$(s_jA + B)^{-1} = [(B + \beta A - (s_j - \beta)(-A)]^{-1}$$
$$= (B + \beta A)^{-1}\left[I + (s_j - \beta)(B + \beta A)^{-1}A\right]^{-1}$$
$$= (B + \beta A)^{-1}\sum_{m=0}^{M_a-1}\left[-(s_j - \beta)A(B + \beta A)^{-1}\right]^m$$

Let $M_a = 1$ and 2, respectively, through the similar previous procedure, and the jth eigenvector derivatives are formulated as

$$d_j = \left\{(B + \beta A)^{-1}\sum_{m=0}^{M_a-1}\left[-(s_j - \beta)A(B + \beta A)^{-1}\right]^m\right.$$
$$+\sum_{k=1,k\neq j}^{N}\left[\left(\frac{s_j - \beta}{s_k - \beta}\right)^{Ma}\frac{\left(\Phi_k\Phi_k^T/a_k\right)}{s_j - s_k}\right.$$
$$\left.+\left(\frac{s_j - \beta}{s_k^* - \beta}\right)^{Ma}\frac{\left(\Phi_k\Phi_k^T/a_k\right)^*}{s_j - s_k^*}\right]$$
$$\left.+\left(\frac{s_j - \beta}{s_j^* - \beta}\right)^{Ma}\frac{\left(\Phi_j\Phi_j^T/a_j\right)^*}{s_j - s_j^*}\right\}f_j + d_{jj} \tag{19}$$

Let β satisfy

$$\left|\frac{s_j - \beta}{s_k - \beta}\right| < 1, \qquad k = 1, 2, \ldots, N \qquad k \neq j$$

For $\Phi_j^T f_j = 0$, the first series is convergent. Since

$$\left|\frac{s_j - \beta}{s_k - \beta}\right|^{Ma} < \left|\frac{s_j}{s_k}\right|^{Ma}, \quad \text{when} \quad k > j$$

Table 1 Relationships between each method and equation

β	M_a	$C = 0$ (no damping)	$C \neq 0$ (viscous damping or non-self-adjoint)
0	0	Ref. 1	Refs. 12 and 13, Eq. (14)
	1	Refs. 3 and 4	Ref. 14, Eq. (16)
	>1	Refs. 8–10	Ref. 14, Eq. (18)
$\neq 0$	=1	Ref. 7	Eq. (19)
	>1		Eq. (19)

the convergence of the second series is further speeded up. Thus, to calculate d_j by Eq. (19), the accuracy will be increased, and the time consumed would be decreased. Effects of inertia (and damping) have been included since the second modal acceleration procedure in the multiple modal accelerations method without shifted-poles. However, in the method with shifted-poles the effects have been included since the first modal acceleration procedure. So, the convergence is speeded up by shifting.

Discussion on Several Aspects

1. Relationships Between Each Method

Equation (19) is a general equation. Based on the values of M_a, β, and C, we would conduct all available modal superposition methods. Table 1 exhibits the relationships between each method presented in the different references and equations in this paper.

2. Simplified Equation

At the s plane, if there are no other poles in the region of the pole s_j, and β is approximately equal to s_j, Eq. (19) can be approximately rewritten as

$$d_j = (B + \beta A)^{-1}\left[I - (s_j - \beta)A(B + \beta A)^{-1}\right]f_j + d_{jj} \qquad (20)$$

In this case, to calculate the derivatives of the jth eigenvector, only the jth eigensolution is required. The computation cost in Eq. (20) is mainly spent for calculating $(B + \beta A)^{-1}$, which is slightly larger than that for solving $2N$ linear equations, and which is approximately equal to the cost by Nelson's method.

3. Determination of an Appropriate Shift Value β

When more than one eigenvector derivative is of interest, only one same shift value β for all eigenvector derivatives is recommended. If the derivatives of mode N_1 to mode N_2 are required, the shift value can be decided as follows:

$$\begin{cases} \beta = \left(\sum_{i=N_1}^{N_2} s_i\right) \Big/ (N_2 - N_1 + 1) \\ \beta \neq s_i, \quad i = N_1, N_1 + 1, \ldots, N_2 \end{cases} \qquad (21)$$

With one shift value for all eigenvector derivatives, the inverse matrix of $(B + \beta A)$ in Eq. (19) can be calculated only once. However, with Nelson's method, the inverse matrices of $(B + s_j A)$ have to be calculated for each eigenvector. Therefore, the presented method would be more efficient than Nelson's method when more than one eigenvector derivatives is required. With a same shift value, Eq. (19) also converges faster than Eq. (18), and so the presented method would be better than Akgun's method in accuracy and efficiency.

4. Error Estimation for Ignoring Modes

Using Eq. (19) to calculate the jth eigenvector derivative, we would only use several modes nearby the jth mode. The effects of ignoring the lower order and higher order modes will be examined in the following. Let d_{jT} be the exact value of the derivative calculated with full modes, and d_{ja} the derivative calculated with the mode N_L to mode N_U, $N_L \leq j \leq N_U$; then the norm of the error between d_{ja} and d_{jT} can be written as

$$\begin{aligned} \|d_{jT} - d_{ja}\|_M &= \left\| \sum_{\substack{k=1 \text{ to } N_L-1 \\ k=N_U+1 \text{ to } N}} \left[\left(\frac{s_j - \beta}{s_k - \beta}\right)^{Ma} \frac{\Phi_k \Phi_k^T / a_k}{s_j - s_k} + \left(\frac{s_j - \beta}{s_k^* - \beta}\right)^{Ma} \frac{\left(\Phi_k \Phi_k^T / a_k\right)^*}{s_j - s_k^*} \right] \right\|_M \\ &\leq \text{Max}\left[\left(\left|\frac{s_j - \beta}{s_{N_L-1} - \beta}\right|\right)^{Ma}, \left(\left|\frac{s_j - \beta}{s_{N_U+1} - \beta}\right|\right)^{Ma} \right] \|d_{jT}\|_M \end{aligned} \qquad (22)$$

The relative error can be written as

$$\begin{aligned} \delta &= \frac{\|d_{jT} - d_{ja}\|_M}{\|d_{jT}\|_M} \\ &\leq \text{Max}\left[\left(\left|\frac{s_j - \beta}{s_{N_L-1} - \beta}\right|\right)^{Ma}, \left(\left|\frac{s_j - \beta}{s_{N_U+1} - \beta}\right|\right)^{Ma} \right] \end{aligned} \qquad (23)$$

If we don't ignore the lower modes, then the relative error becomes

$$\delta \leq \left(\left|\frac{s_j - \beta}{s_{N_U+1} - \beta}\right|\right)^{Ma} \qquad (24)$$

When δ is the allowable margin of the error, we can find the smallest N_U to satisfy Eq. (24).

Numerical Examples

To illustrate the procedure, two discrete mass-stiffness-damping systems, which have 10 and 20 DOF, respectively, are analyzed. The elements of the mass, stiffness, and damping matrices are determined as follows:

$$\begin{cases} m(i,i) = 1 \\ k(i,i) = 20{,}000 + 1000 * i \\ c(i,i) = 2 + 5/i \end{cases} \qquad i = 1, 2, \ldots, N$$

$$\begin{cases} k(i, i-1) = -10{,}000 \\ k(i-1, i) = -10{,}000 \\ c(i, i-1) = -1 \\ c(i-1, i) = -1 \end{cases} \qquad i = 2, 3, \ldots, N$$

where N is the DOF of the system. Tables 2 and 3 list the poles, modal frequencies, and damping ratios of the 10-DOF system and the 20-DOF system, respectively. The eigenvalues are rather closely spaced, which would simulate the cases in aeronautic and astronautic systems. The derivatives of the third and the first three eigenvectors with respect to the stiffness between DOF 5 and ground are calculated. For the different values of M_a and β, the numbers of required

Table 2 Poles, modal frequencies, and damping ratios of the 10-DOF system

Mode no.	Poles Real	Poles Image	Frequncies, Hz	Damping, %
1	−1.0526	70.6869	11.2514	1.489
2	−0.8953	93.7148	14.9159	0.955
3	−1.0710	112.2769	17.8702	0.954
4	−1.3673	131.5218	20.9335	1.040
5	−1.6777	150.6094	23.9717	1.114
6	−1.9766	168.2528	26.7801	1.175
7	−2.2378	183.5569	29.2162	1.219
8	−2.4081	195.9152	31.1832	1.229
9	−2.3804	205.4435	32.6995	1.159
10	−2.2556	214.4697	34.1358	1.052

Table 3 Poles, modal frequencies, and damping ratios of the 20-DOF system

Mode no.	Poles Real	Poles Image	Frequencies, Hz	Damping, %
1	−1.0511	70.6768	11.2498	1.487
2	−0.8335	93.1344	14.8234	0.895
3	−0.7662	107.9167	17.1759	0.710
4	−0.7440	119.3376	18.9935	0.623
5	−0.7472	128.8146	20.5018	0.580
6	−0.7954	137.1824	21.8336	0.580
7	−0.9157	145.2729	23.1214	0.630
8	−1.0793	153.5870	24.4447	0.703
9	−1.2527	162.0783	25.7963	0.773
10	−1.4254	170.5110	27.1386	0.836
11	−1.5945	178.6618	28.4360	0.892
12	−1.7566	186.3442	29.6589	0.943
13	−1.9032	193.3993	30.7820	0.984
14	−2.0107	199.7150	31.7872	1.007
15	−2.0411	205.3552	32.6849	0.994
16	−2.0125	210.7107	33.5372	0.955
17	−1.9921	216.2239	34.4146	0.921
18	−1.9966	222.1465	35.3571	0.899
19	−2.0189	228.7388	36.4063	0.883
20	−2.0577	236.6441	37.6645	0.869

Table 4 Number of required modes for $\delta < 1\%$ in 10-DOF system

Mode no.	M_a	Required modes for $\delta < 1\%$: $\beta = 0$	$\beta = 0.99s_j$	$\beta = (s_1 + s_2 + s_3)/3$
	0	10	—	—
	1	9	2	9
	2	9	1	5
1	3	5	1	3
	4	5	1	2
	5	3	1	2
	0	10	—	—
	1	9	3	3
	2	9	1	1
2	3	5	1	1
	4	5	1	1
	5	5	1	1
	0	10	—	—
	1	10	4	9
	2	10	1	5
3	3	9	1	5
	4	9	1	4
	5	6	1	2

eigenvectors to find the derivatives whose relative errors are less than 1% are shown in Tables 4 and 5. From Tables 4 and 5, we can find the following:

1) Many modes are required for the normal modal superposition-method ($\beta = 0$, $M_a = 0$), otherwise mode truncation errors will be significant.

2) More modes are required for the modal acceleration with shifting ($\beta = 0$, $M_a \geq 1$, Akgun's method[14]). There are mode truncation errors when a smaller value of M_a and few modes are used, or there are roundoff errors when a larger value of M_a is used. To decrease the roundoff errors and truncation errors, many modes are required. The key problem is that the first series in Eq. (18) isn't convergent.

3) Not a few modes are required for one modal acceleration with shifting ($\beta > 0$, $M_a = 1$, similar to the method of Ma and Hagiwara[7]). Especially in the case of one shift value for more than one eigenvector derivative, many modes are also required for further decreasing the truncation errors.

4) Very few modes are required for the multiple modal accelerations method with shifting ($\beta > 0$, $M_a > 1$). When $M_a > 2$ and different shift value ($\beta = 0.99s_j$) for each derivative, only one eigenvector that is being considered is required. In practical situations, choosing $\beta \approx s_j$, the satisfied results would be obtained by Eq. (20). Using a same shift value for three derivatives, and selecting $M_a = 2$ or 3, only few modes are required.

Table 5 Number of required modes for $\delta < 1\%$ in 20-DOF system

Mode no.	M_a	Required modes for $\delta < 1\%$ errors: $\beta = 0$	$\beta = 0.99s_j$	$\beta = (s_1 + s_2 + s_3)/3$
	0	17	—	—
	1	15	2	15
	2	15	1	8
1	3	7	1	4
	4	7	1	4
	5	3	1	4
	0	17	—	—
	1	16	3	4
	2	16	1	3
2	3	9	1	1
	4	9	1	1
	5	7	1	1
	0	17	—	—
	1	16	4	15
	2	16	1	8
3	3	12	1	4
	4	12	1	4
	5	8	1	4

Table 6 Relative cost and required modes (in parentheses) for $\delta < 1\%$

DOFs / Deriv. of mode	$N = 10$: 3	$N = 10$: 1, 2, 3	$N = 20$: 3	$N = 20$: 1, 2, 3
Nelson's	100 (1)	300 (3)	100 (1)	300 (3)
$\beta = 0, M_a = 0$ (modal method)	9.0 (10)	27 (10)	5.0 (17)	15 (17)
$\beta = 0, M_a = 3$ (Akgun's method)	123 (9)	140 (9)	123 (12)	133 (12)
$\beta > 0, M_a = 1$ (Ma's method)	113 (4)	127 (9)	120 (4)	127 (15)
$\beta > 0, M_a = 3$ (presented method)	113 (1)	136 (5)	120 (1)	129 (4)

In several previous cases, the computation cost is much different from each other. Table 6 shows the relative cost and the number of required modes for $\delta < 1\%$ relative errors. The cost for calculating eigensolutions isn't included. From Table 6, comparing the cost and the required modes, the multiple modal accelerations with shifting method is most efficient. Especially for large-scale systems, its efficiency would be more marked.

Conclusions

This paper analyzes the convergence of the modal superposition methods for calculating eigenvector derivatives in a viscous damping system and provides effective strategies to speed up the convergence and to reduce the more truncation errors. A multiple modal accelerations and shifted-poles method is presented and implemented. The results of sample problems are reported and show the effectiveness of the method. The proposed method can greatly improve the accuracy and has the least computation cost within a given margin of accuracy compared with available modal superposition methods and Nelson's method when more than one eigenvector derivative is of interest.

The presented method can be applied directly in systems with zero eigenvalues. Although the method is developed to calculate eigenvector derivatives, it can be used to improve response calculations and substructure syntheses.

References

[1]Fox, R. L., and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[2]Adelman, H. M., and Haftka, R. T., "Sensitivity Analysis of Discrete Structural Systems," *AIAA Journal*, Vol. 24, No. 5, 1986, pp. 823–832.

[3]Sutter, T. R., Camarda, C. J., Walsh, J. L., and Adelman, H. M., "Comparison of Several Methods for Calculating Vibration Mode Shape Derivatives," *AIAA Journal*, Vol. 26, No. 12, 1988, pp. 1506–1511.

[4]Wang, B. P., "Improved Approximate Methods for Computing Eigenvector Derivatives in Structural Dynamics," *AIAA Journal*, Vol. 29, No. 6, 1991, pp. 1018–1020.

[5]Andrew, A. L., "Convergence of an Iterative Method for Derivatives of Eigensystems," *Journal of Computational Physics*, Vol. 26, No. 1, 1978, pp. 107–112.

[6]Ting, T., "Accelerated Subspace Iteration for Eigenvector Derivatives," *AIAA Journal*, Vol. 30, No. 8, 1992, pp. 2114–2118.

[7]Ma, Z. D., and Hagiwara, I., "Sensitivity Analysis Methods for Coupled Acoustic–Structural Systems, Part I: Modal Sensitivity," *AIAA Journal*, Vol. 29, No. 11, 1991, pp. 1787–1795.

[8]Yao, M. B., "Structural Sensitivity Analysis and Design Parameters Modification," M.S. Thesis, Dept. of Aircraft Engineering, Nanjing Univ. of Aeronautics and Astronautics, Nanjing, PRC, 1993.

[9]Chen, S. H., Liu, Z. S., and Zhao, Y. Q., "An Accurate Modal Superposition Method for Computing Mode Shape Derivatives in Structural Dynamics," *Journal of Chinese Society of Theoretical and Applied Mechanics*, Vol. 25, No. 5, 1993, pp. 427–434 (in Chinese).

[10]Zhang, L. M., He, B. Q., and Yuan, X. L., "A Truncated Iterative Modal Method for Design Sensitivity Analysis," *Journal of Nanjing Univ. of Aeronautics and Astronautics*, Vol. 26, No. 3, 1994, pp. 25–33.

[11]Hu, H. C., *Vibration Theory of Multiple DOF Structures*, Academic Press, Beijing, PRC, 1987 (in Chinese).

[12]Rogers, L. C., "Derivative of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 8, No. 5, 1970, pp. 943, 944.

[13]Kuang, J. H., and Chen, L. S., "A Sensitivity Analysis on the Eigen Solutions of Linearly Damped System," *Proceedings of 6th International Modal Analysis Conference*, Kissimmee, FL, 1988, pp. 572–578.

[14]Akgun, M. A., "New Family of Modal Methods for Calculating Eigenvector Derivatives," *AIAA Journal*, Vol. 32, No. 2, 1994. pp. 379–386.

[15]Maddox, N. R., "On the Number of Modes Necessary for Accurate Response and Resulting Forces in Dynamic Analysis," *Journal of Applied Mechanics, ASME*, Vol. 42, No. 2, 1975, pp. 516–571.